NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS

2018 HSC Course Assessment Task 3
(Trial Examination)
Wednesday June 27, 2018.

General Instructions

- Working time –3 hours
  (plus 5 minutes reading time).
- Write using blue or black pen.
  Diagrams may be sketched in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Attempt all questions.

Section I - 10 marks

- Mark your answers on the answer sheet provided.

Section II – 90 marks

- Commence each new question on a new page.
- Show all necessary working in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: ………………………………………… **# BOOKLETS USED:** ……….

Class (please ✓)

☐ Mr Berry
☐ Ms Ziaziaris
☐ Mr Hwang
☐ Mr Zuber
☐ Mr Lin

Question	MC	11	12	13	14	15	16	**Total**
Marks	$\overline{10}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{100}$

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Section 1: Multiple Choice– 1 mark each.

Q1. Which line is perpendicular to the line $4x + 3y + 2 = 0$?

(A) $4x + 3y - 2 = 0$

(B) $4x - 3y + 2 = 0$

(C) $3x + 4y + 2 = 0$

(D) $3x - 4y - 2 = 0$

Q2. The value of $\sum_{k=2}^{20} 10 - 3k$ is

(A) -342

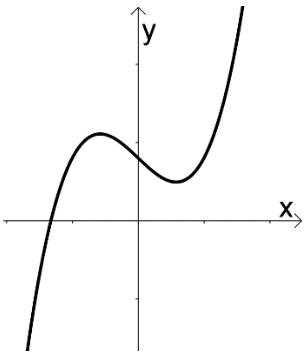
(B) -414

(C) -437

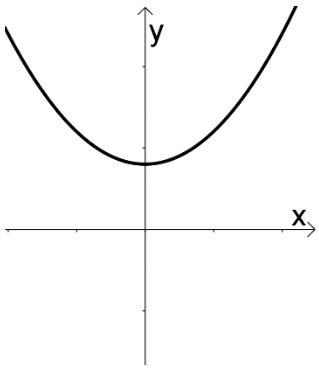
(D) -500

Q3. Which function is an odd function?

(A)



(B)



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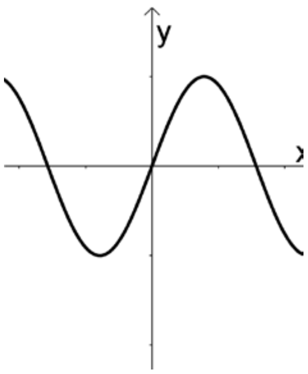

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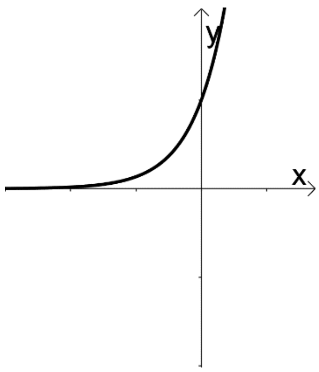


Q4. Using the trapezoidal rule with 4 subintervals, which expression gives the approximate area under the curve $y = x\, e^{x}$ between $x = 1$ and $= 3$?

(A) $\frac{1}{4}(e + 3e^{1.5} + 4e^{2} + 5e^{2.5} + 3e^{3})$

(B) $\frac{1}{4}(e + 6e^{1.5} + 4e^{2} + 10e^{2.5} + 3e^{3})$

(C) $\frac{1}{2}(e + 3e^{1.5} + 4e^{2} + 5e^{2.5} + 3e^{3})$

(D) $\frac{1}{2}(e + 6e^{1.5} + 4e^{2} + 10e^{2.5} + 3e^{3})$

Q5. What is the period of $= 3\tan(4x)$?

(A) $\frac{\pi}{8}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{2}$

(D) $\frac{2\pi}{3}$

Q6. The solution to $3x^2 + 2x > 8$ is

(A) $-\frac{4}{3} < x < 2$

(B) $x < -\frac{4}{3}, x > 2$

(C) $x < -2, x > \frac{4}{3}$

(D) $-2 < x < \frac{4}{3}$

Q7. If α and β are roots of the equation $2x^2 - 4x - 1 = 0$, what is the value of $\alpha^2 + \beta^2$?

.

(A) 3

(B) 4

(C) 5

(D) None of the above

Q8. It is known that $\ln 3a = \ln b - 2\ln c$, where $a, b, c > 0$.

Which statement is true?

(A) $a = \dfrac{b - c^2}{3}$

(B) $a = \dfrac{b}{3c^2}$

(C) $\ln 3a = \dfrac{b}{c^2}$

(D) $\ln 3a = \dfrac{\ln b}{\ln c^2}$

Q9. Evaluate $\displaystyle\int_0^6 |x - 2| \, dx$

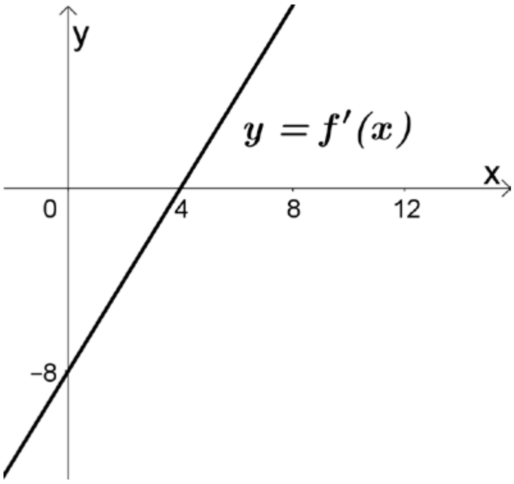
(A) 10

(B) 20

(C) 30

(D) None of the above.

Q10. The graph of $y = f'(x)$ is shown.



The curve $y = f(x)$ is tangential to the x-axis.

What is the equation of the curve $= f(x)$?

(A) $y = 2x^2 - 8x + 8$

(B) $y = 2x^2 - 8x + 16$

(C) $y = x^2 - 8x + 8$

(D) $y = x^2 - 8x + 16$

End of Section I

Section II – Short Answer 90 marks

Question 11 (15 marks) Commence on a NEW page. Marks

(a) Rationalise the denominator $\dfrac{1-\sqrt{5}}{6+\sqrt{5}}$ 2

(b) Factorise fully $16-4x^2$. 2

(c) State the domain of the function $y=\sqrt{4-x}$. 1

(d) Solve $|\,2x-1\,| < 4$. 2

(e) Differentiate $y=5x^6-\sqrt{x}$. 2

(f) Differentiate $y=(\cos x - x)^3$. 2

(g) Find $\int (3x+1)^4\,dx$. 2

(h) Solve $\cos\left(\dfrac{x}{2}\right)=\dfrac{\sqrt{3}}{2}$ for $0 \le x \le 4\pi$. 2

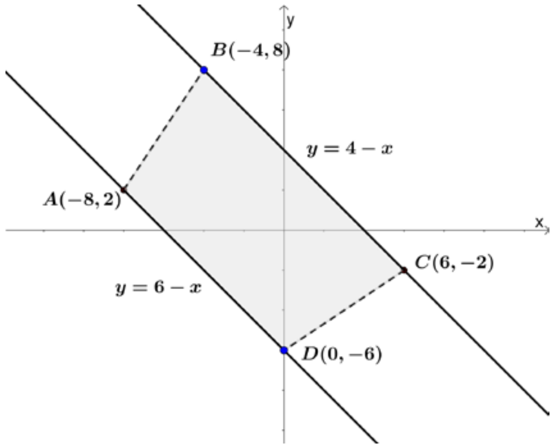
End of Question 11

Question 12 (15 marks) Commence on a NEW page. Marks

(a) The points $A\,(-8,2),\ B(-4{,}8),\ C(6,-2),\ D(0-6)$ define a trapezium in the Cartesian plane.

The equation of the line BC is $y = 4 - x$, and of line AD is $y = 6 - x$.

The distance AD is $8\sqrt{2}$ units.



i) Find the perpendicular distance from the point A to the line BC. 2

ii) Hence calculate the area of the trapezium. 2

(b) Differentiate $y = \log_e \dfrac{3x-1}{(x+1)^4}$ 2

(c) Find:

i) $\int x\,(3 - \sqrt{x}\,)\,dx$ 2

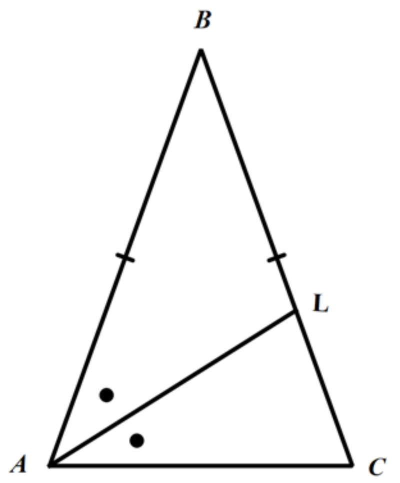
ii) $\int \sin(5x+2)\ dx$ 2

iii) $\int \dfrac{x-5}{x^2-10x}\,dx$ 2

Question 12 continues on the next page.

Question 12 (continued)

(d) In a triangle ABC, $AB = BC$. The point L is on BC such that AL bisects $\angle BAC$.



i) Copy the diagram into your workbook.

ii) If $AL = AC$, find the size of $\angle ABC$, giving reasons. 3

High quality setting out is required for full marks.

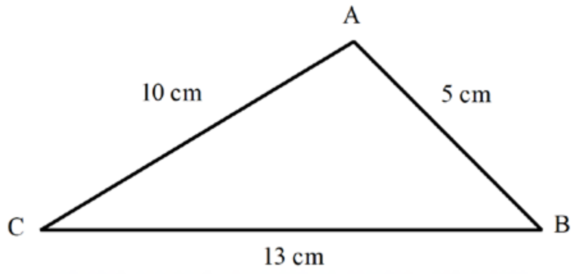
End of Question 12

Question 13 (15 marks) Commence on a NEW page. Marks

(a) State the location of the vertex and the focus of the parabola 3

$$8x - y^2 + 6y - 1 = 0.$$

(b) Triangle ABC has sides AB = 5 cm, BC = 13 cm and $AC = 10$ cm. 3

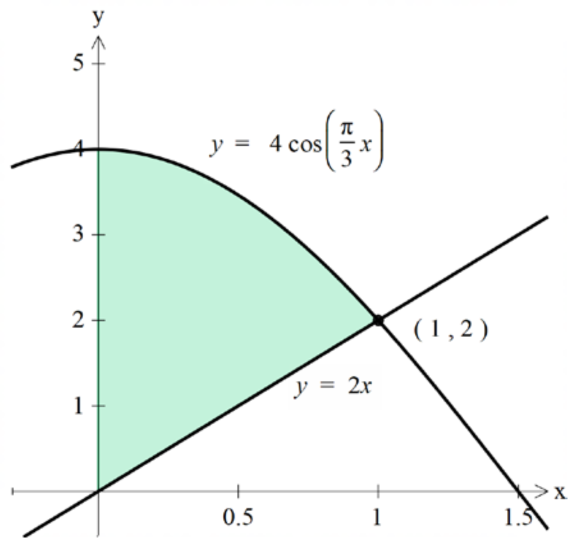


Find the exact value of $\tan C$ in simplest form.

(c) Sketch the region $y \le \sqrt{25 - x^2}$ and $y < x$. 3

(d) Find the values of k for which $y = 5x^2 + (20 - k)x + 20$ is positive definite. 3

(e) The curve $y = 4\cos\left(\frac{\pi}{3}x\right)$ meets the line $y = 2x$ at the point $(1,2)$ as shown in the diagram below.



Find the exact value of the shaded area. 3

End of Question 13

Question 14 (15 marks) Commence on a NEW page. Marks

(a) Given

$$f(x) = (x+2)(x-2)^3$$

and

$$f'(x) = 4(x-2)^2(x+1) = 4x^3 - 12\,x^2 + 16,$$

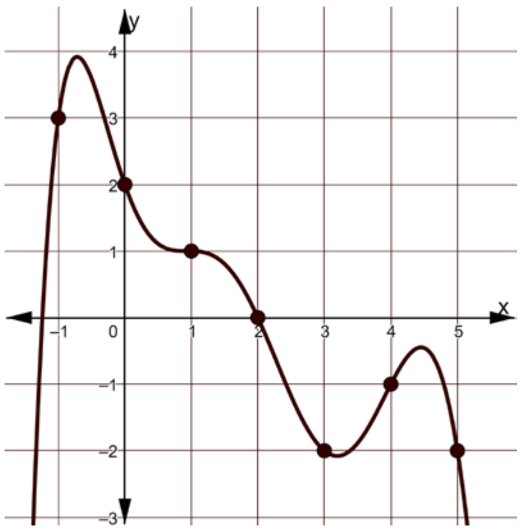
i) Find the stationary points of $y = f(x)$ and determine their nature. 3

ii) Find the coordinates of any points of inflexion. 2

iii) Sketch the graph of $y = f(x)$, clearly indicating the intercepts, stationary points and points of inflexion. 2

(b) Given that $x+y,\ x-y,\ xy$ form an arithmetic sequence, write an expression for x in terms of y. 2

(c) Given the graph of $y = f(x)$ below, 3

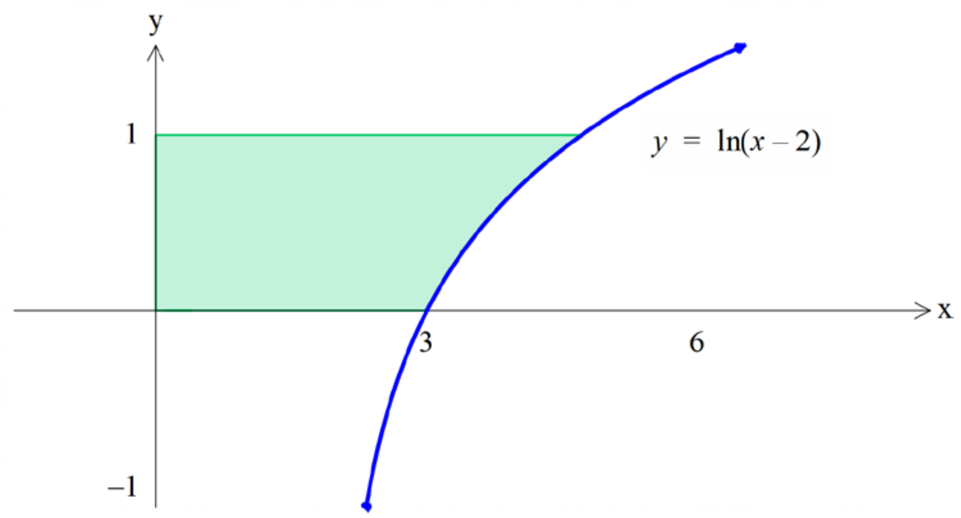


Apply Simpson's Rule with five values to find an approximation for $\int_0^4 f(x)\,dx$.

Question 14 continues on the next page

(d) A region is defined by the function $y = \log_e(x - 2)$, the y −axis, and the lines $y = 0$ and $y = 1$. 3

Find the volume of the solid of revolution formed by rotating the region about the y-axis.



End of Question 14

Question 15 (15 marks) Commence on a NEW page. Marks

(a) The velocity of a particle travelling along the x-axis is given by the equation

$$v = 3 - \frac{12}{2+t}$$

where t is the time in seconds and the velocity is in m/s.

i) When is the particle stationary? 1

ii) What happens to the velocity as $t \to \infty$? 1

iii) Sketch the graph of $v(t)$ for $t \geq 0$, showing any intercepts. 1

iv) Find the acceleration when the particle is stationary. 2

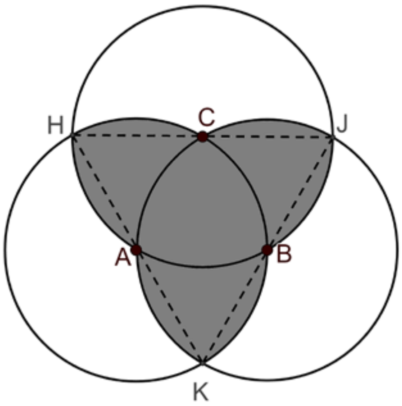
v) Find the distance travelled in the first 6 seconds. 3

(b) i) Show that $\frac{d}{dx}\, x^2 e^{-x^2} = 2xe^{-x^2} - 2x^3 e^{-x^2}$ 2

ii) Hence find $\int x^3\, e^{-x^2}\, dx$ 2

Question 15 continues on the next page

(c) Three circles of radius 1 unit with centres A, B and C respectively are arranged as shown in the diagram below.



i) Find the exact value of the area of the triangle HJK. 1

ii) Hence or otherwise find the exact value of the shaded area. 2

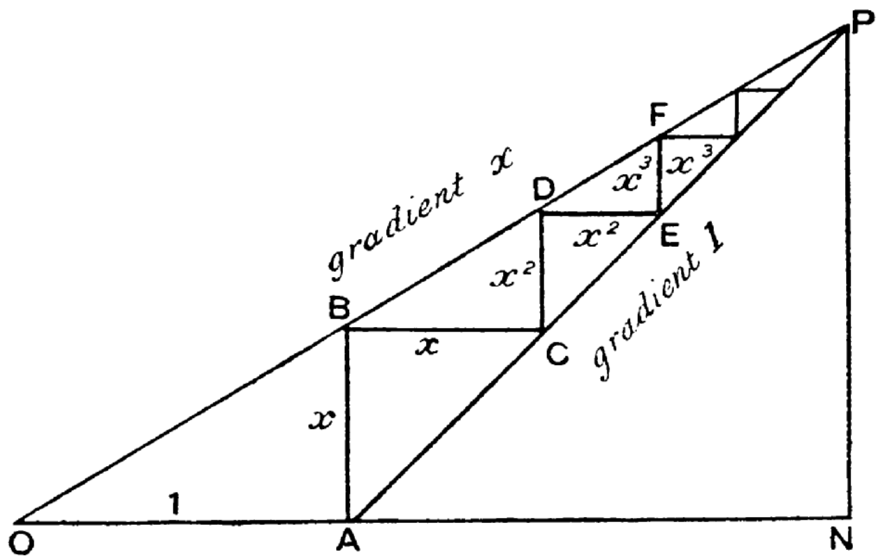
End of Question 15

Question 16 (15 marks) Commence on a NEW page.

(a) By using the property $x = y^{\log_y x}$, or otherwise, show that **2**

$$a^{\log_b x} = x^{\log_b a}$$

(b) The following diagram appears in the 1913 book "Carslaw's Plane Trigonometry":



This diagram has become a famous "Proof Without Words" for the sum of an infinite geometric series.

In the questions below, the aim is to *prove* the result for the sum of an infinite geometric series, so don't use the result in your working out.

We can see from the diagram that $ON = 1 + x + x^2 + x^3 + \cdots$.

i) Explain why $NP = ON - 1$. **1**

ii) Explain why $NP = x\,ON$. **1**

iii) Using the results (i) and (ii) above, show that, **1**

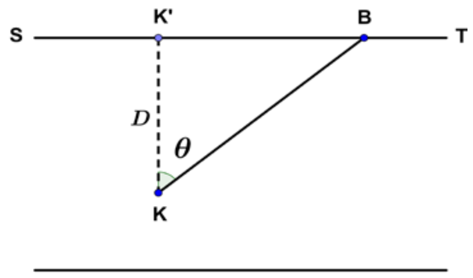
$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

iv) How does this proof demonstrate that x must satisfy the restriction $x < 1$? **1**

Question 16 continues on the next page

(c) At midday, Keanu is in a speedboat on a river at location K when he receives a call from Sandra at location S, riding a bus along a coastal highway towards T.

Sandra asks Keanu to meet up with her bus further along the highway at a location of his choosing – she doesn't mind where they meet, just so long as they eventually meet.



The bus is travelling at a constant speed of V km/hr, scheduled to pass K' at 1 PM. The distance KK' is D km.

Keanu leaves at midday on a bearing of angle θ and meets the bus at point B.

i) Show that the bus arrives at point B at time t hours, 1

$$t = \frac{D \tan\theta}{V} + 1$$

ii) Hence show that Keanu will need to travel at a speed r, where 2

$$r = \frac{D\,V \sec\theta}{D\tan\theta + V}$$

iii) Show that 2

$$\frac{dr}{d\theta} = \frac{DV\sec\theta\;(V\tan\theta - D)}{(D\tan\theta + V)^2}$$

given: $\frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta$

iv) Show that r is minimised when $\tan\theta = \frac{D}{V}$ (reasoning required). 2

v) Show that the minimum speed r is given by $r = \frac{DV}{\sqrt{D^2 + V^2}}$. 2

END OF THE EXAMINATION.

2018 Mathematics HSC Course Task 3

Suggested Responses

1. (D) **2.** (C) **3.** (C) **4.** (A) **5.** (B)
6. (C) **7.** (C) **8.** (B) **9.** (A) **10.** (D)

Question 11 :

(a) (2 marks)

$$\begin{aligned}\frac{1-\sqrt{5}}{6+\sqrt{5}} &= \frac{(1-\sqrt{5})}{6+\sqrt{5}} \times \frac{(6-\sqrt{5})}{(6-\sqrt{5})}\\ &= \frac{(1-\sqrt{5})(6-\sqrt{5})}{36-5}\\ &= \frac{6-\sqrt{5}-6\sqrt{5}+5}{31}\\ &= \frac{11-7\sqrt{5}}{31}\end{aligned}$$

(b) (2 marks)

$$\begin{aligned}16-4x^2 &= 4(4-x^2)\\ &= 4(2+x)(2-x)\end{aligned}$$

(c) (1 mark)

$$x \leq 4$$

(d) (2 marks)

$$\begin{aligned}&-4 \leq 2x-1 \leq 4\\ &-3 \leq 2x \leq 5\\ &-\frac{3}{2} \leq x \leq \frac{5}{2}\end{aligned}$$

(e) (2 marks)

$$\begin{aligned}y &= 5x^6 - x^{\frac{1}{2}}\\ y' &= 30x^5 - \frac{1}{2\sqrt{x}}\end{aligned}$$

(f) (2 marks)

$$\begin{aligned}y &= (\cos x - x)^3\\ y' &= 3(\cos x - x)^2 \times (-\sin x - 1)\\ &= -3(\sin x + 1)(\cos x - x)^2\end{aligned}$$

(g) (2 marks)

$$\begin{aligned}&\int (3x+1)^4 \quad dx\\ &= \frac{(3x+1)^5}{5 \times 3} + C\\ &= \frac{(3x+5)^5}{15} + C\end{aligned}$$

(h) (2 marks)

$$\cos\frac{x}{2} = \frac{\sqrt{3}}{2} \qquad 0 \leq x \leq 4\pi$$

$$0 \leq \frac{x}{2} \leq 2\pi$$

$$\begin{aligned}\frac{x}{2} &= \frac{\pi}{6}, \frac{11\pi}{6}\\ \frac{x}{2} &= \frac{2\pi}{6}, \frac{22\pi}{6}\\ x &= \frac{\pi}{3}, \frac{11\pi}{3}\end{aligned}$$

Question 12 :

(a) i. (2 marks) BC: $x+y-4=0$ A(-8,2)

$$\begin{aligned}d &= \frac{|1(-8)+1(2)-4|}{\sqrt{1^2+1^2}}\\ &= \frac{|-8+2-4|}{\sqrt{2}}\\ &= \frac{10}{\sqrt{2}} = 5\sqrt{2} \text{ units}\end{aligned}$$

ii. (2 marks)

$$\begin{aligned}(BC)^2 &= (-4-6)^2 + (8+2)^2\\ BC &= 2\sqrt{10}\end{aligned}$$

$$\begin{aligned}\text{Area trapezium } &= 5\sqrt{2} \times \frac{8\sqrt{2}+10\sqrt{2}}{2}\\ &= 90 \text{ units }^2\end{aligned}$$

(b) (2 marks)

$$\begin{aligned}y &= \ln\frac{3x-1}{(x+1)^4}\\ &= \ln(3x-1) - 4\ln(x+1)\\ y' &= \frac{3}{3x-1} - \frac{4}{x+1}\end{aligned}$$

(c) i. (2 marks)

$$\int x(3-\sqrt{x})\,dx$$
$$= \int (3x - x^{\frac{3}{2}})\,dx$$
$$= \frac{3}{2}x^2 - \frac{2}{5}x^{\frac{5}{2}} + C$$

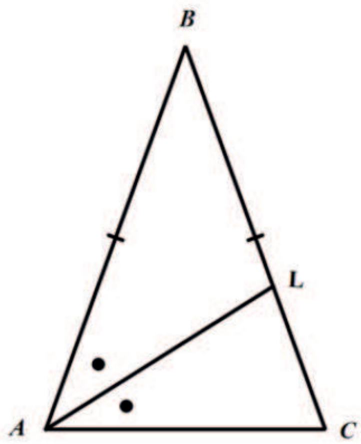
ii. (2 marks)

$$\int \sin(5x+2)\,dx$$
$$= -\frac{1}{5}\cos(5x+2) + C$$

iii. (2 marks)

$$\int x^2 - 10x\,dx = \frac{1}{2}\ln(x^2-10x) + C$$

(d) (2 marks)



Let $\angle BAL = \angle BAC = \theta$ (given)
ΔABC is isosceles (two equal sides)
$\therefore \angle ACB = \angle BAC = 2\theta$ (base angles of isosceles triangle)

ΔALC is isosceles (two equal sides, given)
$\therefore \angle ACB = \angle ALC$
$\therefore \angle ALC = 2\theta$

In ΔALC,
$\angle LAC + \angle ACL + \angle CLA = 180° = 5\theta$ (angle sum of a triangle)
$\therefore \theta = \frac{180}{5} = 36°$

In ΔABC,
$\angle ABC + 2\theta + 2\theta = 180°$ (angle sum of a triangle)
$\therefore \angle ABC = 180° - 2\times 36° - 2\times 36° = 36°$

Question 13 :

(a) (3 marks)

$$8x - y^2 + 6y - 1 = 0$$
$$y^2 - 6y = 8x - 1$$
$$y^2 - 6y + 9 = 8x - 1 + 9$$
$$(y-3)^2 = 8(x+1)$$

(draw a diagram!)
vertex: (-1, 3)
focus: (1, 3)

(b) (3 marks)

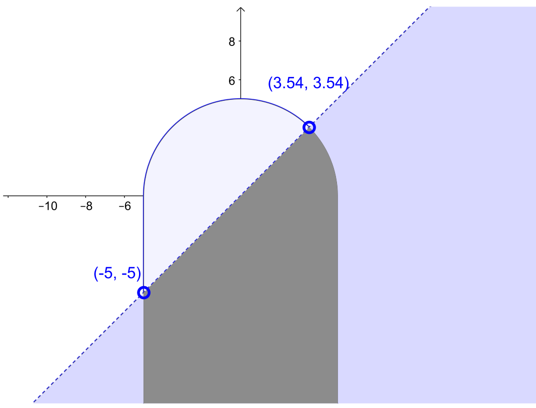
$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$
$$\cos C = \frac{10^2+13^2-5^2}{2\times 10\times 13}$$
$$\cos C = \frac{61}{65}$$

(draw a diagram!)

$$x = \sqrt{65^2 - 61^2} = 6\sqrt{14}$$
$$\tan C = \frac{x}{61}$$
$$\tan C = \frac{6\sqrt{14}}{61}$$

(c) (3 marks)

$$y \le \sqrt{25-x^2} \quad , y \le x$$



(d) (3 marks) $y = 5x^2 + (20-k)x + 20$ is positive definite when $\Delta < 0$

$$(20-k)^2-4(5)(20)<0$$
$$400-40k+k^2-400<0$$
$$k^2-40k<0$$
$$k(k-40)<0$$
(draw a diagram)
$$0<k<40$$

(e) (3 marks)

$$\text{Area} = \int_0^1 (4\cos\frac{\pi}{3}x - 2x)\,dx$$
$$= \left[-\frac{12}{\pi}\sin\frac{\pi}{3}x - x^2\right]_0^1$$
$$= (0-0) - (-\frac{12}{\pi}\sin\frac{\pi}{3} - 1^2)$$
$$= \frac{6\sqrt{3}-\pi}{\pi}\text{ units }^2$$

Question 14 :

(a) $f(x) = (x+2)(x-2)^3$

$f'(x) = 4(x-2)^2(x+1) = 4x^3 - 12x^2 + 16$

i. (3 marks)
Stationary when $f'(x) = 0$
ie: $4(x-2)^2(x+1) = 0$
$x = 2$ or $x = -1$

$f(2) = 0\,, f(-1) = 27$

Stationary points at $(2,0)\,(-1,-27)$

x	-2	-1	0	2	3
$f'(x)$	-64	0	16	0	16
	\	—	/	—	/

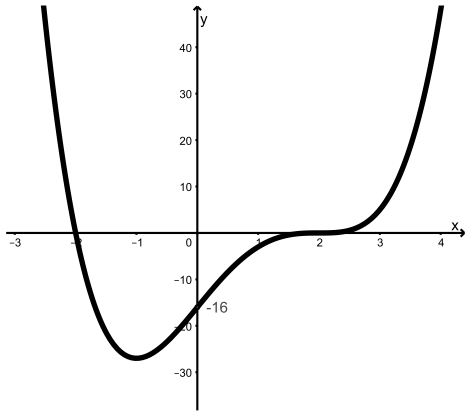
$\therefore$ horizontal point of inflexion at $(2,0)$, local minimum at $(-1,-27)$

ii. (2 marks)
$f''(x) = 12x^2 - 24x$
Possible points of inflection at $f''(x) = 0$ ie: $12x^2 - 24x = 0$
$x(x-2) = 0$
$x = 0$ or $x = 2$
We have already shown $x = 0$ is a point of inflexion, test for $x = 2$

x	1	2	3
$f''(x)$	-12	0	36
	$\cap$	—	$\cup$

$\therefore$ Points of inflexion at $(2,0), (0{-}16)$

iii. (2 marks)



(b) (2 marks) AP: $x+y$, $x-y$, xy
$a = x+y$, $d = x-y-(x+y) = -2y$

$$T_3 - T_2 = T_2 - T_1$$
$$xy-(x-y) = (x-y)-(x+y)$$
$$xy - x + y = x - y - x - y$$
$$xy = -2y + x - y$$
$$xy - x = -2y - y$$
$$x(y-1) = -3y$$
$$x = \frac{3y}{1-y}$$

(c) (3 marks) Using Simpson's rule:

x	0	1	2	3	4
$f(x)$	2	1	0	-2	-1
	$\times$ 1	$\times$ 4	$\times$ 2	$\times$ 4	$\times$ 1

$$\int_0^4 f(x)\,dx \approx \frac{1}{3}(2{+}1{\times}4{+}0{\times}2{+}(-2){\times}4{+}{-}1)$$
$$= -1$$

(d) (3 marks) $V = \pi\int_0^1 x^2 dy$

$$y = \ln(x-2)$$
$$x - 2 = e^y$$
$$x = e^y + 2$$
$$x^2 = (e^y+2)^2$$
$$= e^{2y} + 4e^y + 4$$

$$
\begin{aligned}
V &= \pi \int_0^1 (e^{2y} + 4e^y + 4)\, dy \\
&= \pi \left[\frac{1}{2} e^{2y} + 4e^y + 4y \right]_0^1 \\
&= \pi \left(\frac{1}{2} e^{2(1)} + 4e^{(}1) + 4(1) \right) \\
&\quad - \pi \left(\frac{1}{2} e^{2(0)} + 4e^{(}0) + 4(0) \right) \\
&= \pi \left(\frac{1}{2} e^2 + 4e - \frac{1}{2} \right) \text{ units }^3
\end{aligned}
$$

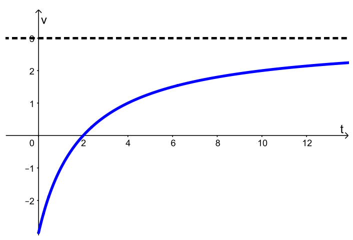
Question 15 :

(a) i. (1 mark) Stationary when $v = 0$

$$
\begin{aligned}
3 - \frac{12}{2+t} &= 0 \\
\frac{12}{2_t} &= 3 \\
3(2+t) &= 12 \\
2 + t &= 4 \\
t &= 2 \text{ seconds}
\end{aligned}
$$

ii. (1 mark) As $t \to \infty\,, v \to 3$ m/s

iii. (1 mark)



iv. (2 marks)

$$
\begin{aligned}
a &= \frac{d}{dt} \left(3 - \frac{12}{2+t} \right) \\
&= \frac{d}{dt} \left(3 - 12(2+t)^{-1} \right) \\
&= 12(2+t)^{-2} \\
&= \frac{12}{(2+t)^2}
\end{aligned}
$$

$$
\begin{aligned}
a(2) &= \frac{12}{(2+2)^2} \\
&= \frac{3}{4} \text{ m/s }^2
\end{aligned}
$$

v. (3 marks)

$$x(t) = 3t - 12 \ln (t+2) + C$$

$$
\begin{aligned}
x(0) &= 3(0) - 12 \ln (0) + 2) + C \\
&= -12 \ln 2 + C
\end{aligned}
$$

$$
\begin{aligned}
x(2) &= 3(2) - 12 \ln (2) + 2) + C \\
&= 6 - 12 \ln 4 + C \\
&= 6 - 24 \ln 2 + C
\end{aligned}
$$

$$
\begin{aligned}
x(6) &= 3(6) - 12 \ln (6) + 2) + C \\
&= 18 - 12 \ln 8 + C \\
&= 18 - 36 \ln 2 + C
\end{aligned}
$$

particle moves left $t = 0$ to $t = 2$:
distance $= x(0) - x(2)$
$= (-12 \ln 2 + C) - (6 - 24 \ln 2 + C)$
$= 12 \ln 2 - 6$

particle moves right $t = 2$ to $t = 6$:
distance $= x(6) - x(2)$
$= (18 - 36 \ln 2 + C) - (6 - 24 \ln 2 + C)$
$= 12 - 12 \ln 2$

Total distance $= (12 \ln 2 - 6) + (12 - 12 \ln 2) = 6$ metres

(b) i. (2 marks) $\frac{d}{dx} \left(x^2 e^{-x^2} \right)$:
Let $u = x^2\,, u' = 2x$
Let $v = e^{-x^2}\,, v' = -2xe^{-x^2}$

$$
\begin{aligned}
\text{LHS} &= \frac{d}{dx} \left(x^2 e^{-x^2} \right) \\
&= u'v + uv' \\
&= 2x \times e^{-x^2} - x^2 \times -2xe^{-x^2} \\
&= 2xe^{-x^2} - 2x^3 e^{-x^2} \\
&= \text{RHS}
\end{aligned}
$$

ii. (2 marks) From (i),

$$\frac{d}{dx}\left(x^2e^{-x^2}\right) = 2x^3e^{-x^2} - 2xe^{-x^2}$$

$$2\int x^3e^{-x^2}dx = \int 2xe^{-x^2}dx - \int\left(\frac{d}{dx}x^2e^{-x^2}\right)dx$$

$$2\int x^3e^{-x^2}dx = \int 2xe^{-x^2}dx - x^2e^{-x^2}$$

$$2\int x^3e^{-x^2}dx = -e^{-x^2} - x^2e^{-x^2} + C \text{ from (i)}$$

$$\int x^3e^{-x^2}dx = -\frac{1}{2}\left(e^{-x^2} + x^2e^{-x^2}\right) + C$$

(c) i. (1 mark)
ΔHJK is an equaliteral triangle, sides 2 units
area $= \frac{1}{2} \times 2 \times 2 \times \sin\pi/3$
$= \sqrt{3}\text{u}^2$

ii. (2 marks)

$$\begin{aligned}\text{segment area } &= \frac{1}{2}r^2(\theta - \sin\theta)\\ &= \frac{1}{2}(\frac{\pi}{3} - \sin\frac{\pi}{3})\\ &= \frac{1}{2}(\frac{\pi}{3} - \frac{\sqrt{3}}{2})\\ &= \frac{2\pi - 3\sqrt{3}}{12}\end{aligned}$$

$$\begin{aligned}\therefore \text{Area} &= \sqrt{3} + 6 \times \frac{2\pi - 3\sqrt{3}}{12}\\ &= \left(\pi - \frac{\sqrt{3}}{2}\right)\text{ u}^2\end{aligned}$$

Question 16 :

(a) (2 marks)

$$\begin{aligned}\text{LHS} &= a^{\log_b x}\\ &= \left(x^{\log_x a}\right)^{\log_b x}\\ &= x^{\log_x a \times \log_b x}\\ &= x^{\frac{\ln a}{\ln x} \times \frac{\ln x}{\ln b}}\\ &= x^{\frac{\ln a}{\ln b}}\\ &= x^{\log_b x}\\ &= \text{ RHS}\end{aligned}$$

(b) i. (1 mark)
AP has gradient 1, so $NP = AN$.
From the diagram, $ON = 1 + AN$
$\therefore ON = 1 + NP$
$\therefore NP = ON - 1$

ii. (1 mark) OP has gradient x,
$\therefore \frac{NP}{ON} = x$

$\therefore NP = xON$

iii. (1 mark)

$$\begin{aligned}ON - 1 &= x\,ON \text{ from (i), (ii)}\\ ON - xON &= 1\\ ON(1-x) &= 1\\ ON &= \frac{1}{1-x}\end{aligned}$$

From the diagram,
$ON = 1 + x + x^2 + x^3 + ...$
$\therefore 1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$

iv. If $x \geq 1$ then the line OB will not intersect the gradient=1 line AP.

or

If $x \geq 1$ then $x^2 > x, x^3 > x^2$, etc, so the sequence of triangles will not reduce in size, so the diagram will not work.

NB: the distances will not become negative if $x \geq 1$ - it is possible to draw this condition, however the lines OB and AP do not intersect.

(c) i. (1 mark)
K'B is $D\tan\theta$
S travel time $= \frac{D\tan\theta}{V}$

Sandra takes 1 hr to reach K', so

$$t = \frac{D\tan\theta}{V} + 1$$

ii. (2 marks)
KB is $D\sec\theta$
Keanu needs to travel this distance

in the time S reaches B

$$\begin{aligned} r &= (D\sec\theta) \div \left(\frac{D\tan\theta}{V} + 1\right) \\ &= (D\sec\theta) \div \left(\frac{D\tan\theta + V}{V}\right) \\ &= \frac{DV\sec\theta}{D\tan\theta + V} \end{aligned}$$

iii. (2 marks)

$$\frac{dr}{d\theta} = \frac{u'v - uv'}{v^2}$$

$$\begin{aligned} uv' - uv' &= (DV\sec\theta\tan\theta)(D\tan\theta + V) \\ &\quad - (D\sec^2\theta)(DV\sec\theta) \\ &= DV\sec\theta(D\tan^2\theta \\ &= \qquad +V\tan\theta - D\sec^2\theta) \\ &= DV\sec\theta(D\tan^2\theta + V\tan\theta \\ &\quad - D(1+\tan^2\theta)) \\ &= DV\sec\theta(V\tan\theta - D) \end{aligned}$$

$$\frac{dr}{d\theta} = \frac{DV\sec\theta(V\tan\theta - D)}{(D\tan\theta + V)^2}$$

iv. (2 marks)

Stationary points when $\frac{dr}{d\theta} = 0$.

$\sec\theta > 0$ for the domain $0 \le \theta < \frac{\pi}{2}$.

$(V\tan\theta - D) = 0$ when $\tan\theta = \frac{D}{V}$.

Let θ_m be the value for which $\tan\theta_m = \frac{D}{V}$

To show that r is a minimum for θ_m, we must test using the first or second derivative (!).

Consider $\alpha < \theta_m$.
Since $\tan\theta$ is an increasing function for $0 \le \theta < \frac{\pi}{2}$,
$\tan\alpha < \tan\theta_m < \frac{D}{V}$
$\therefore (V\tan\alpha - D) < (V\frac{D}{V} - D) < 0$

Consider $\beta > \theta_m$.
Since $\tan\theta$ is an increasing function for $0 \le \theta < \frac{\pi}{2}$,
$\tan\beta > \tan\theta_m > \frac{D}{V}$
$\therefore (V\tan\beta - D) > (V\frac{D}{V} - D) > 0$

θ	α	θ_m	β
$\sec\theta - D$	(+)	(+)	(+)
$V\tan\theta - D$	(-)	0	(+)
$(D\tan\theta + V)^2$	(+)	(+)	(+)
$\frac{dr}{d\theta}$	(+)(-)/(+)	0	(+)(+)/(
	\	—	/

Therefore r is a minimum when $\tan\theta = \frac{D}{V}$.

v. (2 marks)

If $\tan\theta = \frac{D}{V}$, draw diagram
then $\sec\theta = \frac{\sqrt{D^2+V^2}}{V}$, (θ acute).

$$r = \frac{DV\sec\theta}{D\tan\theta + V}$$

$$r = \frac{DV\frac{\sqrt{D^2+V^2}}{V}}{D(\frac{D}{V}) + V}$$

$$r = \frac{D(\sqrt{D^2+V^2})}{\frac{D^2}{V} + V}$$

$$r = \frac{D(\sqrt{D^2+V^2})}{\frac{D^2+V^2}{V}}$$

$$r = \frac{DV(\sqrt{D^2+V^2})}{D^2+V^2}$$

$$r = \frac{DV}{\sqrt{D^2+V^2}}$$